Maximum Log Likelihood Method

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The quantity being minimized is the negative log likelihood f = -LogLikelihood(x, y, z, t, xang, yang, LogE).

The total LogLikelihood is the sum of SiteLogLikelihood functions, which in turn are the sum of the two ShmooLogLikelihood functions. Two methods are defined for calculating the ShmooLogLikelihood, ShmooLogLikelihood_Primitive (version 3.01), and ShmooLogLikelihood_Integral (version 3.02). Version 3.01 is the one currently in use.

Version 3.01

In ShmooLogLikelihood_Primitive, the likelihood is calculated as the sum of the likelihood of each event at the time t_{event} and the likelihood of the total hit intensity (in MIP) I_{shmoo}:

$$P_{\text{total}} = \prod_{\text{shmoos}} P_{\text{I}}(I_{\text{shmoo}}) \times \prod_{\text{events}} P_{\text{T}}(t_{\text{event}})$$

$$\log P_{\text{total}} = \sum_{\text{shmoos}} \log P_{\text{I}}(I_{\text{shmoo}}) + \sum_{\text{events}} \log P_{\text{T}}(t_{\text{event}})$$

The likelihood function $P_{\text{T}}(t_{\text{event}})$ (called HitLogLikelihood_Primitive in the code) was originally given by the TDF weighted by the intensity of the event:

$$\log P_{\text{T}}(t_{\text{event}}) = \log(I_{\text{event}} \times P_{\text{TDF}}(t_{\text{event}}, r, \theta, \log E))$$

The function P_{TDF} is normalized. There is an optional weight parameter that can be used to scale it, currently set to 1.

Better results were obtained by replacing this function with:

$$\log P_{\text{T}}(t_{\text{event}}) = \log(P_{\text{Poission}}(I_{\text{event}}; I_{\text{LDF}} \times P_{\text{TDF}}(t_{\text{event}}, r, \theta, \log E))$$

The likelihood function $P_{\text{I}}(I_{\text{shmoo}})$ (called IntensityLogLikelihood_Primitive in the code) is given by the probability of measuring the total intensity from a Poisson distribution about the fitted value of the LDF at the site:

$$\log P_{\text{I}}(I_{\text{shmoo}}) = \log(P_{\text{Poisson}}(I_{\text{shmoo}}; I_{\text{LDF}}))$$

Sites with no hits are treated as being consistent with $I_{\text{LDF}} < 0.25$:

$$\log P_{\text{I}}(I_{\text{shmoo}} = 0) = \log(0.5 \times P_{\text{Poisson}}(I_{\text{shmoo}} = 0.25; I_{\text{LDF}}))$$

Version 3.01 with KS Test

This version is similar to Version 3.01, with the HitLogLikelihood_Primitive function replaced by HitLogLikelihood_KSTest, which calculates the Kolmogorov-Smirnov probability of getting the observed time distribution of hits at a given shmoo from the TDF.

$$\log P_{\text{total}} = \sum_{\text{shmoos}} \log P_{\text{I}}(I_{\text{shmoo}}) + \sum_{\text{shmoos}} \log P_{\text{KS}}(t_1, ..., t_n)$$

The probability P_{KS} is a function of the quantity D, which is the maximum difference between two cumulative distributions, in this case the total charge measured at the shmoo before time t, and the integrated TDF:

$$D = \text{Max}\,|S_N(t) - P(t)| \;\; \text{over} \;\; -\infty < t < \infty$$

The quantity $S_N(t)$ is the sum of hits at the shmoo before time t, normalized by the total charge measured at the shmoo:

$$S_N(t) = \sum_{t_n < t} I_{\text{event}}(t_n)/I_{\text{shmoo}}$$

The quantity $P(t)$ is the integral of the TDF from 0 to t:

$$P(t) = \int_0^t P_{\text{TDF}}(t_{\text{event}}, r, \theta, \log E)$$

The probability of D being greater than the observed value is given by

$$P_{\text{KS}}(D > \text{observed}) = Q_{\text{KS}}\left(\left[\sqrt{N} + 0.12 + 0.11/\sqrt{N}\right] D\right)$$

where N is the number of data points, and

$$Q_{\text{KS}} = 2\sum_{j=1}^{\infty}(-1)^{j-1}e^{-2j^2\lambda^2}$$

The function which calculates Q_{KS} in ROOT calculates the number of terms to sum based on the value of λ.

Because the integral of measured charge at a site over time is a step function, the maximum value of D will occur either immediately before or immediately after a hit. These are the only times checked by the code, to avoid having to step through all values of t.

Version 3.02

The function ShmooLogLikelihood_Integral calls the two sub-functions HitLogLikelihood_Integral and NoHitLogLikelihood_Integral (the probability of the observed hits occuring and the probability that there were no additional hits):

$$P_{\text{total}} = \prod_{\text{hits}} P_{\text{HIT}}(I_{\text{event}}, t_{\text{event}}) \times \prod_{\text{shmoos}} P_{\text{NOHIT}}$$

$$\log P_{\text{total}} = \sum_{\text{hits}} \log P_{\text{HIT}} + \sum_{\text{shmoos}} \log P_{\text{NOHIT}}$$

The probability of a given hit with intensity I_{event} occuring at time t_{event} is given by:

$$\log P_{\text{HIT}}(t_{\text{event}}, I_{\text{event}}) = \log P_{\text{Poisson}}(I_{\text{event}}; I_{\text{expected}})$$

where I_{expected} is calculated from the LDF by integrating the TDF at the site over an interval $[-t_{\Delta}, t_{\Delta}]$ around t_{event}:

$$I_{\text{expected}} = \int_{t_{\text{event}}-t_{\Delta}}^{t_{\text{event}}+t_{\Delta}} (I_{\text{LDF}} P_{\text{TDF}} + P_{\text{BACKGROUND}})\, dt$$

Here P_{TDF} is the normalized time distribution function and I_{LDF} is the expected total intensity from the LDF. The quantity t_{Δ} is currently set to 12.5 ns. The level of background noise is set to 200 Hz.

The probability of not observing a hit is given by the probability of observing zero from a Poisson distribution about the expected intensity minus the any observed hits:

$$\log P_{\text{NOHIT}} = \log P_{\text{Poisson}}(0; I'_{\text{expected}})$$

$$I'_{\text{expected}} = \left(I_{\text{LDF}} + \int_{t_{\text{event}}-t_{\max}}^{t_{\text{event}}+t_{\max}} P_{\text{BACKGROUND}}\, dt \right) - \sum_{\text{events}} I_{\text{expected}}(t_{\text{event}})$$

The quantity $t_{\max}$ is currently set to 50 μs.