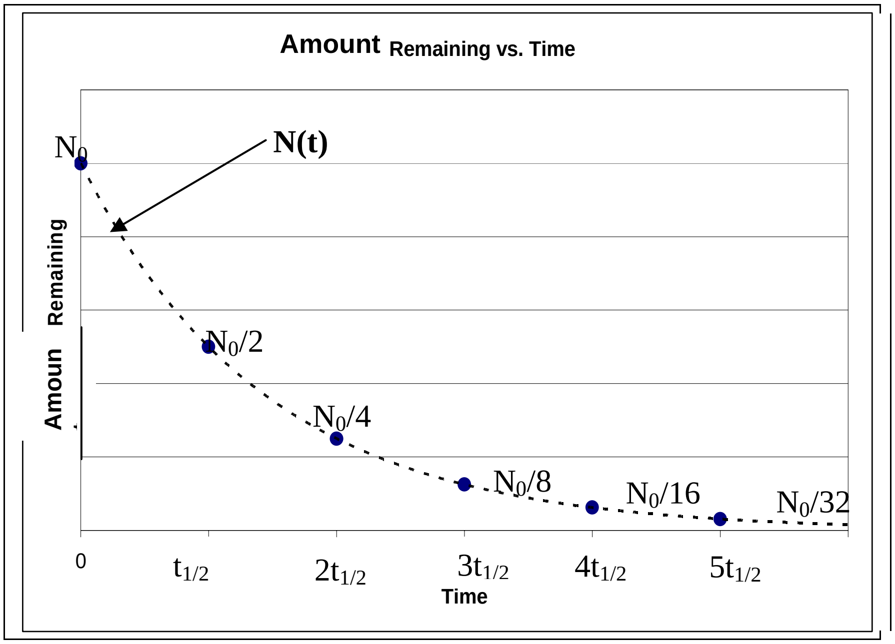
Half-Life and How to Measure It

The following graph shows an exponential decay of radioactive nuclei over time. At time zero, an initial amount N_0 is present in the sample. Solid points show the amount of material remaining after one, two, three, four, or five half-lives. The dotted curve suggests an actual continuous decay over time, described by a mathematical function N(t).



Each individual radioactive nucleus actually decays according to the laws of probability: it has a 50% chance of decaying for every half-life it goes through. When we take a large sample of them, the result of this individual process is that *on average* the number of radioactive nuclei in the sample goes down by a factor of two for every $t_{1/2}$ in time. It's only an average, though, and each particle decays (or not) purely by chance. So in a real experiment, there will be some randomness in the number of particles left over time. For a very large total sample of nuclei, this randomness becomes a very small fraction and is not very noticeable and we don't need to worry about it. Then we would observe the smooth behavior as a function of time shown in the graph shown above.

The dotted curve in the graph is actually described by the following simple function:

$$N(t) = N_0 * 2^{-t/t_{1/2}} = \frac{N_0}{2^{t/t_{1/2}}}.$$

It is easy to verify that for t=n $t_{1/2}$, where n is an integer (0,1,2,3…), we reproduce the sequence of values

N_0, $N_0/2$, $N_0/4$, $N_0/8$, …

Let's now think about these N_0 nuclei decaying in a different way. Each nucleus decays after a specific time, so let's label each nucleus with a number that corresponds to the order in which it decayed. The first nucleus to decay is labeled with a "1", the second with a "2", and so on until the last nucleus which is labeled "N_0". This is a fun but useful game. We can now use these labels to designate the time that each nucleus decays. T_1 is the time when the first nucleus decays, T_2 for the second, etc. So now there are N_0 decay times (one for each nucleus) which we can write as

T_1, T_2, T_3, … T_{N0} .

SIMPLE IDEA: We can divide the list into two equal length lists with $N_0/2$ earlier times and $N_0/2$ later times. The value at the end of the first list, $T_{N0/2}$ and the first value in the second list $T_{(N0/2+1)}$ are good estimates for the half-life $t_{1/2}$! (Maybe take the average of these two values and declare victory!)

BETTER IDEA: Set up a series of bins, each for a fixed time interval Δt, and for each bin count the number of nuclei remaining at that time. Since each bin contains the count of the number of nuclei remaining, we can graph the value of the count in each bin versus the time of the bin and obtain a good approximation to the function N(t) shown in the graph above. Now we need to get the half-life from this graph. The trick is to use the logarithm again. Remember the equation for the number of remaining muons N(t):

$$N(t) = N_0 * 2^{-t/t_{1/2}} = \frac{N_0}{2^{t/t_{1/2}}}.$$

If we take the logarithm of both sides of the equation, we get the following:

$$\log N(t) = \log(N_0) - \frac{t}{t_{1/2}} * \log(2)$$

which has the form of the equation of a straight line:

$$y = mx + b \; .$$

If we make a graph of log[N(t)] vs. t, the slope is given by

$$-\frac{\log(2)}{t_{1/2}}$$

Which means the slope of the line can be used to solve for the half life $t_{1/2}$.